Sharp estimates for Brownian non-intersection probabilities

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Abstract

This paper gives an accessible (but still technical) self-contained proof to the fact that the intersection probabilities for planar Brownian motion are given in terms of the intersection exponents, up to a bounded multiplicative error, and some closely related results. While most of the results are already known, the proofs are somewhat new, and the paper can serve as a source for the estimates used in our paper [10] on the analyticity of the Brownian intersection exponents.

1 Introduction

In a recent series of papers [7, 8, 9, 10], the authors rigorously derived the values for the intersection exponents for planar Brownian motion. Among other things, we prove in these papers that the Hausdorff dimension of the outer boundary of a planar Brownian path is $4/3$ (see [11] for an overview). This paper is complementary to these papers in that it proves some estimates about the Brownian intersection probabilities that do not depend on knowing their exact values.

The intersection exponents are defined as the asymptotic rate of decay of certain non-intersection probabilities. The main results in this paper give estimates for the actual probabilities in terms of these asymptotic exponents.

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For example, it is easy to show by subbadditivity that the probability that the paths of two independent planar Brownian motions started from uniform-independent points on the unit circle will not intersect before hitting the circle of radius R is $R^{-\xi+o(1)}$ as $R \to \infty$ (this is the definition of the exponent $\xi = \xi(1,1)$). We show that in fact the probability is equal to $u(R)R^{-\xi}$, where $u(R) \in [1/c, c]$ for some constant c independent of $R \geq 1$.

This, and most other results proven here have been derived before by Lawler (see [4] and reference therein). However, these earlier treatments were a little complicated at times (one reason is that they simultaneously treated both the planar and three-dimensional cases) and hence it seems worthwhile to have a self-contained account of these results. We will not make any use of our recent papers [7, 8, 10, 9]; instead this paper can be considered as a prerequisite to [10]. The results presented here are used in [10] to prove analyticity of the mappings $\lambda \mapsto \xi(k,\lambda)$ on $(0,\infty)$. "Up-to-constants" estimates are also instrumental in relating the intersection exponent to the Hausdorff dimension of exceptional sets of the Brownian path, see, e.g., [3].

We will concentrate on the intersection exponents $\xi(2,\lambda)$ which are relevant for analyzing the outer boundary of Brownian paths. However, the proofs, with only minor changes, adapt easily to other Brownian intersection exponents (see Section 7).

For all $r \geq 0$, let $\mathcal{C}_r$ denote the circle of radius $\exp(r)$ about zero. Let Y^0, Y^1, Y^2 be independent planar Brownian motions starting at 0. Define for $j = 0, 1, 2$, and $r \in \mathbb{R}$,

$$T_r^j = \inf\{t > 0 : Y_t^j \in \mathcal{C}_r\}$$

and the paths

$$\mathcal{Y}_r^j = Y^j[T_0^j, T_r^j]$$

(one could equivalently have taken Brownian motions started uniformly on the unit circle up to their hitting time of $\mathcal{C}_r$). We define the random variable (depending on $\mathcal{Y}_r^1$ and $\mathcal{Y}_r^2$),

$$Z_r = Z_r(\mathcal{Y}_r^1, \mathcal{Y}_r^2) := \mathbf{P}[\mathcal{Y}_r^0 \cap (\mathcal{Y}_r^1 \cup \mathcal{Y}_r^2) = \emptyset \mid \mathcal{Y}_r^1, \mathcal{Y}_r^2].$$

This is the probability, given $\mathcal{Y}_r^1$ and $\mathcal{Y}_r^2$, that another Brownian motion started uniformly on the unit circle reaches $\mathcal{C}_r$ without intersecting the paths $\mathcal{Y}_r^1$ and $\mathcal{Y}_r^2$. We define for all $\lambda > 0$,

$$a_r = a_r(\lambda) = \mathbf{E}[(Z_r)^\lambda].$$

Note that when λ is an integer, then a_r is the probability that λ independent copies of $\mathcal{Y}_r^0$ do not intersect $\mathcal{Y}_r^1 \cup \mathcal{Y}_r^2$. It is straightforward to show that there exists a constant ξ, usually denoted by $\xi(2, \lambda)$, such that

$$\lim_{r \to \infty} (a_r)^{1/r} = e^{-\xi}.$$

One of the main goals of the present paper is to present a short and self-contained proof of estimates for a_r (and alternative closely related quantities) up to multiplicative constants. In particular:

Theorem 1.1. *For every $\lambda_0 > 0$, there exist constants $0 < c_1 < c_2 < \infty$ such that for every $0 < \lambda \leq \lambda_0$ and every $r \geq 2$,*

$$c_1 e^{-r\xi(2,\lambda)} \leq a_r(\lambda) \leq c_2 e^{-r\xi(2,\lambda)}.$$

This theorem is a slight improvement over the estimate given for a_r in [4]. In that paper, it was shown that for every $0 < \lambda_1 < \lambda_2 < \infty$, one can find constants c_1, c_2 that work for all $\lambda \in [\lambda_1, \lambda_2]$. The approach we give in this paper gives the stronger result that the constants can be chosen uniformly in $(0, \lambda_0]$. An advantage of Theorem 1.1 is that the following is an easy corollary obtained by fixing r and letting $\lambda \to 0+$.

Corollary 1.2. *There exist constants $0 < c_1 < c_2 < \infty$ such that*

$$c_1 e^{-r\xi(2,0)} \leq \mathbf{P}[Z_r > 0] \leq c_2 e^{-r\xi(2,0)},$$

where $\xi(2, 0) := \lim_{\lambda \to 0+} \xi(2, \lambda)$.

Note that $Z_r > 0$ means that $\mathcal{Y}_r^1 \cup \mathcal{Y}_r^2$ does not disconnect $\mathcal{C}_0$ from $\mathcal{C}_r$. This corollary was derived in [3] for the disconnection exponent ξ_0 defined by $e^{-\xi_0} = \lim_{r \to \infty} \mathbf{P}[Z_r > 0]^{1/r}$. However, a more complicated argument was needed [4] to prove $\xi_0 = \lim_{\lambda \to 0+} \xi(2, \lambda)$. Using Theorem 1.1, this is immediate.

Although we do not prove it in this paper, it can actually be shown that quantities like $e^{r\xi(2,\lambda)} a_r$ approach a limit (see the end of Section 6).

Another goal of the present paper is to clarify and summarize the equivalence between the definitions of the exponents in terms of Brownian excursions, Brownian motions, extremal distance, and discuss the influence of the starting points, etc. In fact, we will first focus on another quantity b_r defined in terms of Brownian excursions and extremal distance, show up-to-constants estimates for b_r and then deduce the estimates for a_r.

2 Preliminaries

Before studying non-intersection probabilities, we first review a few easy facts concerning Brownian excursions and extremal distance.

Throughout the paper, for all $r < r'$, $\mathcal{C}_r$ will denote the circle of radius $\exp(r)$ about 0, and $\mathcal{A}(r, r')$ will denote the open annulus between $\mathcal{C}_r$ and $\mathcal{C}_{r'}$. $\mathcal{D}(z, \delta)$ will denote the open disk of radius δ about z. It will be sometimes more convenient to work in the cylindrical metric. We will then implicitely use the fact that for all $\varepsilon > 0$, when δ is sufficiently small, for all $z = e^u \in \mathcal{C}_r$,

$$\mathcal{D}(z, \delta e^r(1-\varepsilon)) \subset \{e^v : |v - u| < \delta\} \subset \mathcal{D}(z, \delta e^r(1+\varepsilon)).$$

2.1 Excursion measure and conformal invariance

Let Y be a Brownian motion starting at the origin, let T_r be its hitting time of the circle $\mathcal{C}_r$ and define

$$S_r = \sup\{t < T_r : Y_t \in \mathcal{C}_0\}.$$

The paths

$$B_t := Y_t, \quad S_r \leq t \leq T_r, \tag{2.1}$$

are called "Brownian upcrossings" of the annulus $\mathcal{A}(0, r)$. We will not care about the time-parameterization of the upcrossings; in particular, it does not matter if the 'starting-time' of the upcrossings is called 0 or S_r.

This probability measure on Brownian upcrossings is very closely related to the Brownian excursion measure that we used in the papers [5, 6, 7, 8]. The excursion measure on the annulus $\mathcal{A}(0, r)$ is the upcrossing probability normalized so that the total mass is $2\pi r^{-1}$.

We now briefly recall some of the properties of these measures. First, there are various equivalent ways of defining them. Define the excursion measure on $\mathcal{A}(0, r)$ starting at $z \in \mathcal{C}_0$ by

$$\mu_{z,r} = \epsilon^{-1} \lim_{\epsilon \to 0} \mu_{z,r,\epsilon}$$

where $\mu_{z,r,\epsilon}$ is the measure obtained from starting a Brownian motion at $(1+\epsilon)z$, killing it upon leaving $\mathcal{A}(0, r)$, and restricting to those paths that exit $\mathcal{A}(0, r)$ at $\mathcal{C}_r$. Then the excursion measure on $\mathcal{A}(0, r)$ is given by

$$\int_0^{2\pi} \mu_{\exp(i\theta),r} \, d\theta. \tag{2.2}$$

Yet another equivalent way to define the probability measure on upcrossings is to identify upcrossings with the process $R_t = \exp(U_t^1 + iU_t^2)$ where U^1 is a three-dimensional Bessel process started at 0, and U^2 an independent Brownian motion started uniformly on $[0, 2\pi]$, stopped at the first time it hits the circle $\mathcal{C}_r$ (i.e., at the first time U^1 hits r) (see e.g. [13] for definition and properties of Bessel processes).

When $r < r'$, define the excursion measure and the upcrossing probability on $\mathcal{A}(r, r')$ as the measure (or law) of e^r times a Brownian upcrossing in $\mathcal{A}(0, r' - r)$. It is easy to see (for instance using the definition of the upcrossings in terms of Bessel processes) that if B is a Brownian upcrossing of $\mathcal{A}(r, r')$, then the time-reversal of $1/B$ is a Brownian upcrossing of $\mathcal{A}(-r', -r)$.

One can in fact define excursion measures in any open planar domain. In the papers [5, 6, 7, 8] we used Brownian excursion measures in simply connected planar domains. Just as in [5], in the present paper, we will need to use this measure only in some particular simply connected domains. Suppose O is a simply connected subset of $\mathcal{A}(r, r')$, and define $\partial_1 := \partial O \cap \mathcal{C}_r$ and $\partial_2 := \partial O \cap \mathcal{C}_{r'}$. Let Φ denote a conformal map from O onto the unit disk. We say that O is a path domain in $\mathcal{A}(r, r')$ if $\Phi(\partial_1)$ and $\Phi(\partial_2)$ are two arcs of positive length. We call ∂_3 and ∂_4 the two other parts of ∂O (possibly viewed as sets of prime ends). The excursion measure in O can be defined as the excursion measure in $\mathcal{A}(r, r')$ restricted to those upcrossings that stay in O.

An important property of the excursion measure is its conformal invariance: if F is a conformal transformation taking a path domain O to another path domain O' in such a way that $F(\partial_1) = \partial_1'$ and $F(\partial_2) = \partial_2'$ (with obvious notation) then the image of the excursion measure on O by F is the excursion measure on O'. See for instance [5, 6] for a proof of this fact.

2.2 Extremal distance and excursions

For any path domain O, there exists a unique positive real L such that O can be mapped conformally onto the half-annulus $O'_L = \{\exp(u + i\theta) : 0 < u < L \text{ and } 0 < \theta < \pi\}$ in such a way that ∂_1 and ∂_2 are mapped onto the semi-circles (or equivalently, such that O can be mapped conformally onto the rectangle $\mathcal{R}_L := (0, L) \times (0, \pi)$ in such a way that ∂_1 and ∂_2 are mapped onto the vertical sides of $\mathcal{R}_L$). We call $L = L(O)$ the π-extremal distance between ∂_1 and ∂_2 in O. This is π times the extremal distance as in [1, 12].

The excursion measure can also be defined on the rectangle $\mathcal{R}_L$ by taking image of the excursion measure in O'_L under the logarithmic map. Alternatively, it can directly be defined as π/ε times the limit when $\varepsilon \to 0$ of the law of Brownian paths started uniformly on the segment $[\varepsilon, \varepsilon + i\pi]$ and restricted to the event where they exit the rectangle through $[L, L + i\pi]$.

Since the excursion measure is invariant under conformal transformations, its total mass depends only on L. By considering directly excursions in the rectangle, it is easy to check that there exists a constant c such that for all $L \geq 1$, the total mass of the excursion measure in $\mathcal{R}_L$ is in $[c^{-1}e^{-L}, ce^{-L}]$. In other words, up to multiplicative constants, $e^{-L(O)}$ measures the total mass of the excursion measure in O.

Extremal distance in a simply connected domain O can be defined in a more general context. For instance (see, e.g., [1]), suppose that V_1 and V_2 are arcs on the boundary of O, and let Γ denote the set of (smooth) paths that disconnect V_1 from V_2 in O. For any piecewise smooth metric ρ in O, define the ρ-area $\mathcal{A}_\rho(O) := \int_O \rho(x+iy)^2 \, dx \, dy$ of O and the length of smooth curves γ, $\ell_\rho(\gamma) := \int_\gamma \rho(z) \, d|z|$. Then, define

$$L(O; V_1, V_2) := \pi \inf_\rho \mathcal{A}_\rho(O)$$

where the infimum is taken over the set of piecewise smooth metrics such that for all $\gamma \in \Gamma$, $\ell_\rho(\gamma) \geq 1$. It is straightforward to check that this definition generalizes the previous one (it is also invariant under conformal transformations, and if V_1 and V_2 are the vertical sides of $O = \mathcal{R}_L$, the infimum is obtained for a constant $\rho = 1/\pi$).

Using rectangles, it is easy to see that this definition is equivalent to the more usual definition (see [1]) of extremal distance in terms of the family of curves connecting V_1 to V_2 in O (i.e. $L(O; V_1, V_2)$ is the maximum over all metrics ρ with $\mathcal{A}_\rho(O) = 1$ of the square of the ρ-distance between V_1 and V_2 in O).

It is straightforward to see that $L(O; V_1, V_2)$ satisfies monotonicity relations: if $O' \subset O, \partial'_1 \subset \partial_1$, and $\partial'_2 \subset \partial_2$, then $L(O'; \partial'_1, \partial'_2) \geq L(O; \partial_1, \partial_2)$; and if C is a simple curve in O connecting ∂_3 and ∂_4, O' is the connected component of $O \setminus C$ whose boundary contains ∂_1, and O^* is the component of $O \setminus C$ whose boundary contains ∂_2, then $L(O; \partial_1, \partial_2) \geq L(O'; \partial_1, C) + L(O^*; C, \partial_2)$.

2.3 A few simple lemmas

We will need a few simple technical facts about extremal distance. It will be more convenient here to work with the cylindrical metric. Let $\tilde{O}$ be a path domain on $\mathcal{A}(0,r)$ (with $\tilde{\partial}_1, \dots, \tilde{\partial}_4$ being the four parts of $\partial\tilde{O}$). Throughout this section, we will use a simply connected set O such that $\exp(O) = \tilde{O}$. We define $\partial_1, \dots, \partial_4$ the parts of O corresponding to $\tilde{\partial}_1, \dots, \tilde{\partial}_4$, and we will suppose that ∂_3 is 'below' ∂_4 (i.e. that $z_1 := \partial_3 \cap \{\Re(z) = 0\}$ lies below $z_2 := \partial_4 \cap \{\Re(z) = 0\}$). Note that $\partial_3 \cap \partial_4 = \emptyset$ (while it was possible that $\tilde{\partial}_3 \cap \tilde{\partial}_4 \neq \emptyset$). The following lemmas will be formulated in terms of O, and applied later to $\tilde{O} = \exp(O)$. We will not bother to choose optimal constants as only their existence will be needed.

Lemma 2.1. *Suppose that for some $\delta < r - 1$, $\mathcal{D}(z_1, 4\delta) \cap \partial_4 = \emptyset$ and $\mathcal{D}(z_2, 4\delta) \cap \partial_3 = \emptyset$. Then,*

$$L(O \setminus [\mathcal{D}(z_1,\delta) \cup \mathcal{D}(z_2,\delta)]; \partial_1, \partial_2) \leq L(O; \partial_1, \partial_2) + 6\pi^2.$$

Proof. Let O' be the domain $O \setminus [\mathcal{D}(z_1,\delta) \cup \mathcal{D}(z_2,\delta)]$ and write $\partial'_1 = [z_1 + i\delta, z_2 - i\delta], \partial'_2 = \partial_2, \partial'_3, \partial'_4$ for the corresponding boundaries. Let ρ be the extremal metric for finding the length of the collection Γ of curves in O connecting ∂_3 and ∂_4 (note that ρ is the conformal image of a multiple of the Euclidean metric in the rectangle, and therefore ρ is smooth) so that

$$\mathcal{A}_\rho(O) = \pi^{-1} L(O; \partial_1, \partial_2).$$

If we let Γ' be the collection of curves in O' connecting ∂'_3 and ∂'_4, and

$$\rho' = \max\{\rho, \delta^{-1}[1_{\mathcal{D}(z_1,2\delta)} + 1_{\mathcal{D}(z_2,2\delta)}]\}$$

in O', then every curve in Γ' has length at least one in the metric ρ'. Hence

$$\begin{aligned} L(O'; \partial'_1, \partial'_2) &\leq \pi \mathcal{A}_{\rho'}(O') \\ &\leq \pi[\mathcal{A}_\rho(O) + 2(4\pi - \pi)] \\ &= L(O; \partial_1, \partial_2) + 6\pi^2. \quad \square \end{aligned}$$

Lemma 2.2. *For all $\delta > 0$, there exists $c(\delta)$ such that if $V \subset \partial_1$ is a segment of length at least δ, if $\operatorname{dist}(V, \partial_3 \cup \partial_4) > \delta$ and if the δ-neighborhood of V disconnects ∂_3 from ∂_4 in $O \cap ((0,\delta) \times \mathbb{R})$, then*

$$L(O; V, \partial_2) \leq L(O; \partial_1, \partial_2) + c(\delta).$$

Proof. Let ρ denote the extremal metric in O associated to $L(O;\partial_1,\partial_2)$ (i.e., any path from ∂_3 to ∂_4 in O has ρ-length at least one, and $\mathcal{A}_\rho(O)$ is minimal), and define

$$\rho' = \max\{\rho, \delta^{-1}1_{(0,\delta)\times\mathbb{R}}].$$

Any path disconnecting V from ∂_2 has ρ'-length at least one, so that $L(O;V,\partial_2) \leq \pi\mathcal{A}_{\rho'}(O)$ and the lemma follows. $\square$

Lemma 2.3. *Suppose that $1 < s < r-1$, and that for some small δ, $\partial_3' := \partial_3 \cap ((s-\delta,s+\delta)\times\mathbb{R})$ and $\partial_4' := \partial_4 \cap ((s-\delta,s+\delta)\times\mathbb{R})$ are both of diameter smaller than $\delta^{1/6}$ and at distance at least $\delta^{1/7}$ from each other. Let V denote the segment in $O \cap \{Re(z)=s\}$ that disconnects ∂_1 from ∂_2 (it is unique because of the previous conditions). Then, for some $C(\delta)$,*

$$L(O;\partial_1,\partial_2) \leq L(O\cap((0,s)\times\mathbb{R});\partial_1,V) + L(O\cap((s,r)\times\mathbb{R});V,\partial_2) + C(\delta).$$

Proof. Let O_1 and O_2 denote the sets $O\cap((0,s)\times\mathbb{R})$ and $O\cap((s,r)\times\mathbb{R})$. Let ρ_1 (resp., ρ_2) denote the extremal metric in O_1 associated to $L(O_1;\partial_1,V)$ (resp., in O_2 associated to $L(O_2;V,\partial_2)$). Let $\mathcal{V} = O\cap((s-\delta,s+\delta)\times\mathbb{R})$. Note that (since $\exp O = \tilde{O}$) the euclidean area of $\mathcal{V}$ is at most $4\pi\delta$. Define

$$\rho = \max(\rho_1,\rho_2,(1/\delta)1_{\mathcal{V}}).$$

It is easy to check that any path joining ∂_3 to ∂_4 in O has ρ-length at least 1 (either, it stays in one of the three sets O_1, O_2 or $\mathcal{V}$, or it contains a path joining $\{\Re(z)=s\}$ to $\{|\Re(z)-s|=\delta\}$). Therefore,

$$L(O;\partial_1,\partial_2) \leq \pi\mathcal{A}_\rho(O) \leq L(O_1;\partial_1,V) + L(O_2;V,\partial_2) + C(\delta). \quad \square$$

2.4 Extending excursions

Let $0 < r < r'$. A consequence of the strong Markov property of planar Brownian motion and of the second definition of the Brownian excursion measure is that if B is a Brownian upcrossing of $\mathcal{A}(0,r)$ defined under the excursion measure, and if one starts from its endpoint (on $\mathcal{C}_r$) another independent planar Brownian motion killed at its first hitting of $\mathcal{C}_{r'}$, restricted to the event that it does not intersect $\mathcal{C}_0$ (note that this is an event of probability r/r'), then the concatenation of the upcrossing with the Brownian path is exactly defined under the Brownian excursion measure in $\mathcal{A}(0,r')$.

In particular, this shows that if B is an Brownian upcrossing of $\mathcal{A}(0,r')$ (defined under the probability measure on upcrossings), then it can be decomposed into two parts: A Brownian upcrossing of $\mathcal{A}(0,r)$ and a Brownian motion started from the end-point of the first part, conditioned to hit $\mathcal{C}_{r'}$ before $\mathcal{C}_0$.

This can also be formulated easily in terms of the definition of Brownian upcrossings using three-dimensional Bessel processes. In particular, it shows that it is possible to define on the same probability space a process $(R_t, t \geq 0)$ started uniformly on the unit circle, such that for each $r > 0$, the process R stopped at its hitting time T_r of the circle $\mathcal{C}_r$ is a Brownian upcrossing of $\mathcal{A}(0,r)$. We will use the σ-field $\mathcal{F}_r$ generated by $(R_t, t \leq T_r)$ in Section 6.

Another simple consequence of the strong Markov property of planar Brownian motion is the fact that conditionaly on $Y(S_r)$ (which has uniform law on $\mathcal{C}_0$), the Brownian upcrossing $Y[S_r, T_r]$ is independent from the initial part $Y[0, S_r]$. Consider now the event $H = H_r$ that $Y[T_0, S_r]$ does not contain a closed loop about zero contained entirely in the annulus $\mathcal{A}(-1,0)$. This event is independent of the upcrossing $Y[S_r, T_r]$ so that on this event, the measure on upcrossings is the same as the upcrossing probability or the excursion measure except that it has a slightly different normalization constant i.e., its total mass m_r is the probability of H_r. We claim there is a constant c such that $c^{-1}r^{-1} \leq m_r \leq cr^{-1}$. The lower bound can for instance be derived by considering the event $\{Y[T_0, T_r] \cap \mathcal{A}(-1,0) \subset \mathcal{D}(Y(T_0), \delta)\}$ for some fixed $\delta < 1/4$. For the upper bound, let k denote the total number of times the Brownian motion goes from $\mathcal{C}_0$ to $\mathcal{C}_{-1}$ before time T_r. Every time the path goes from $\mathcal{C}_0$ to $\mathcal{C}_{-1}$ there is a positive probability, say ρ of forming a closed loop in $\mathcal{A}(-1,0)$. From this and the strong Markov property, we get $\mathbf{P}(H_r \cap \{k = l\}) \leq (1-\rho)^l r^{-1}$, and summing over l gives the upper bound.

We note that we have just proved that for all $\delta < 1/4$, there is a $c' = c'(\delta)$ such that conditioned on the event H_r, the probability that $Y[T_0, T_r] \cap \mathcal{A}(-1,0) \subset \mathcal{D}(Y(T_0), \delta)$ is at least c'.

3 Lower bound

From this point on, we fix a λ_0 and consider $\lambda \in (0, \lambda_0]$. Constants are allowed to depend on λ_0 but not on λ.

Suppose that B^1 and B^2 are two independent Brownian upcrossings of the annulus $\mathcal{A}(0,r)$ defined using the Brownian motions Y^1 and Y^2. Let O^1

and O^2 be the components of $\mathcal{A}(0,r) \setminus (B^1 \cup B^2)$ which are at zero distance from $\mathcal{C}_r$. We choose O^1 in such a way that it has the positively oriented arc on $\mathcal{C}_r$ from the endpoint of B^1 to the endpoint of B^2 as part of its boundary. For $j = 1, 2$, let $L_r^j = L(O^j)$ be π times the extremal distance between $\mathcal{C}_0 \cap \partial O^j$ and $\mathcal{C}_r \cap \partial O^j$ in O^j. Note that $\mathcal{C}_0 \cap \partial O^j$ is a.s. either empty or an arc, and $\mathcal{C}_r \cap \partial O^j$ is a.s. an arc (note that in this case O^j is a.s. a path domain). When $\partial O^j \cap \mathcal{C}_0 = \emptyset$, set $L_r^j := \infty$. Let $L_r := \min\{L_r^1, L_r^2\}$, and let $O := O^j$ when $L_r = L_r^j < \infty$. Define

$$b_r = b_r(\lambda) = r^{-2}\mathbf{E}[\exp(-\lambda L_r)].$$

The goal of the next two sections is to define the intersection exponent $\xi(2,\lambda)$ in terms of b_r, and to prove the following estimates for b_r.

Theorem 3.1. *For any $\lambda > 0$, there exists $\xi(2,\lambda) \in (0,\infty)$ such that $e^{-\xi(2,\lambda)} = \lim_{r\to\infty} b_r^{1/r}$. Furthermore, for any $\lambda_0 > 0$, there exist constants c_1 and c_2 such that for all $\lambda \in (0, \lambda_0]$, and for all $r \geq 0$,*

$$c_1 e^{-r\xi(2,\lambda)} \leq b_r(\lambda) \leq c_2 e^{-r\xi(2,\lambda)}.$$

In the present section, we will derive the lower bound and the next section will be devoted to the (harder) upper bound, and we will relate a_r to b_r in the subsequent section. Note that b_r is decreasing in r because of the monotonicity properties of extremal distance.

For any positive integer n, let E_n denote the event that neither $Y^1[T_0^1, T_n^1]$ nor $Y^2[T_0^2, T_n^2]$ hit the circle $\mathcal{C}_{-1}$. Note that $\mathbf{P}(E_n) = 1/(n+1)^2$ and that E_n is independent from $Y^1[S_n^1, T_n^1]$ and $Y^2[S_n^2, T_n^2]$. Hence

$$E[e^{-\lambda L_n} 1_{E_n}] = n^2 b_n/(n+1)^2.$$

We call $b_n^{\#}$ this quantity.

Lemma 3.2. *There exists a constant c such that for all $n, m \geq 1$,*

$$b_{m+n+1} \leq c b_m b_n.$$

Proof. First consider the event $H_n^1 \cap H_n^2$ that neither $Y^1[T_0^1, T_n^1]$ nor $Y^2[T_0^2, T_n^2]$ contains a closed loop in $\mathcal{A}(-1,0)$ that surrounds $\mathcal{C}_{-1}$. The previous considerations show that L_n is independent from $H_n^1 \cap H_n^2$ so that

$$b_n^* := \mathbf{E}[e^{-\lambda L_n} 1_{H_n^1 \cap H_n^2}] \leq \mathbf{P}[H_n^1 \cap H_n^2] \mathbf{E}[e^{-\lambda L_n}] \leq c b_n. \tag{3.1}$$

Once we have this, to get the lemma we split the upcrossings into the pieces up to T^j_m and from T^j_{m+1} to T^j_{m+n+1}. Monotonicity of extremal distance gives

$$b^{\#}_{m+n+1} \leq c b^{\#}_m b^*_n,$$

from which the lemma follows. $\square$

Using this lemma, we can now define $\xi(2,\lambda)$ by $e^{-\xi(2,\lambda)} = \lim_{n\to\infty} b_n^{1/n}$ and get $b_n \geq c e^{-n\xi(2,\lambda)}$ for some c, which gives the lower bound in Theorem 3.1 for integer n's. By considering Brownian motions restricted to stay in the upper or lower half-plane we get the crude estimate $\xi(2,\lambda) \leq 2+\lambda \leq 2+\lambda_0$. We will use this fact implicitely in our estimates when we write $e^{-\xi(2,\lambda)} \geq c$. This is obvious, but it is important that the constant can be chosen uniformly for $0 < \lambda \leq \lambda_0$. In this case $c = e^{-(2+\lambda_0)}$ suffices. In particular, since b_r is decreasing in r, it suffices to prove the theorem for integer values of r.

In Section 4.1 we will need the following lemma. Since the proof is very similar to that of (3.1) we include it here. If $\epsilon > 0$, let $E_{n,\epsilon}$ be the event that neither Brownian motion hits $\mathcal{C}_{-1+\epsilon}$ before reaching $\mathcal{C}_n$.

Lemma 3.3. *There is a constant c such that for every $\epsilon \in (0, 1/4)$,*

$$\mathbf{E}\left[\, e^{-\lambda L_n}\,(1_{E_n} - 1_{E_{n,\epsilon}})\,\right] \leq c\epsilon b^{\#}_n.$$

Proof. First note that $E_n \setminus E_{n,\epsilon}$ is independent of L_n so that the left-hand side is equal to $\mathbf{P}[E_n \setminus E_{n,\varepsilon}]\mathbf{E}[e^{-\lambda L_n}]$. Moreover, $\mathbf{P}[E_n] = 1/(n+1)^2$ and $\mathbf{P}[E_{n,\epsilon}] = (1-\varepsilon)^2/(n+1-\varepsilon)^2 \geq (1-c\varepsilon)\mathbf{P}[E_n]$. $\square$

4 The upper bound

Our goal in this section is to derive the upper bound in Theorem 3.1. It suffices to give an upper bound for $\tilde{b}_n := n^{-2}\mathbf{E}[\exp(-\lambda L^1_n)]$ since $\tilde{b}_n \leq b_n \leq 2\tilde{b}_n$. The basic strategy is to find a sequence $(b^\delta_n)_{n\geq 1}$ such that:

- For all $n \geq 1$, $b^\delta_n \leq \tilde{b}_n$.
- There is a c_1 such that for all $n, m \geq 1$,

$$b^\delta_{n+m+2} \geq c_1 b^\delta_n b^\delta_m. \tag{4.1}$$

- For all $n \geq 1$,

$$\#\{j \in \{1, \ldots, n\} : b_j^\delta \geq \tilde{b}_j/2\} \geq 3n/4. \tag{4.2}$$

Suppose we find such a sequence b_n^δ. It is then easy to check that $\lim_{n\to\infty}(b_n^\delta)^{1/n} = \lim_{n\to\infty}(b_n)^{1/n} = e^{-\xi(2,\lambda)}$, and using (4.1), that there is a constant c_3 such that for all $n \geq 1$,

$$b_n^\delta \leq c_3 e^{-n\xi(2,\lambda)}.$$

Also, (4.2) implies that for each $n \geq 2$ there is a $j \in \{1, \ldots, n-1\}$ such that

$$b_j^\delta \geq b_j/2, \quad b_{n-j}^\delta \geq b_{n-j}/2,$$

and hence

$$\tilde{b}_{n+2} \leq b_{n+1} \leq c b_j b_{n-j} \leq 4c b_j^\delta b_{n-j}^\delta \leq 4cc_1^{-1} b_{n+2}^\delta \leq 4c_3 cc_1^{-1} e^{-\xi(2,\lambda)(n+2)}.$$

This establishes the upper bound.

4.1 Nice configurations

Throughout this section, we will use Brownian upcrossings B^1 and B^2 of annuli $\mathcal{A}(r, r')$. For convenience, we use the convention that B^j is started at time zero on $\mathcal{C}_r$, and that $T_{r''}^j$ denotes the first time at which B^j hits $\mathcal{C}_{r''}$.

We define a class of "nice" configurations for pairs of Brownian upcrossings B^1, B^2 of $\mathcal{A}(r, r')$ for $r' - r > 1$. More precisely, we say that the configuration is δ-nice at the beginning if:

- $L^1 < \infty$;
- $d(B^1(0), B^2(0)) > \delta^{1/8} e^r$.
- For all $\eta < \delta$, $B^j[0, T_{r+\eta^{1/2}}^j] \subset \mathcal{D}(B^j(0), \eta^{1/4} e^r)$ for $j = 1, 2$.
- For all $\eta < \delta$, $B^j[T_{r+\eta^{1/2}}^j, T_{r+1}] \cap \mathcal{A}(r, r + 4\eta) = \emptyset$ for $j = 1, 2$.
- $B^j[T_{r+1}^j, T_{r'}^j] \cap \mathcal{A}(r, r + 4\delta) = \emptyset$ for $j = 1, 2$.

Here we write $L^1 = L^1(r, r')$ for the appropriate π-extremal distance. Note that (and this is the reason for which we introduce conditions with $\eta < \delta$) if a domain is δ-nice at the beginning, then it is δ'-nice at the beginning for any $\delta' < \delta$.

Note also that the second, third and fourth conditions are only on $B^1[0, T^1_{r+1}]$ and $B^2[0, T^2_{r+1}]$. If we use $U(\delta)$ to denote the event that all these three conditions hold, then, as the law of $B^j[0, T^j_{r+1}]$ is that of a Brownian upcrossing of $\mathcal{A}(r, r+1)$, we get easily that

$$\mathbf{P}[U(\delta)] \to 1 \tag{4.3}$$

as $\delta \to 0+$, uniformly in $r' > r+1$. In particular, almost surely, the configuration of a pair of Brownian upcrossings is δ-nice at the beginning for sufficiently small δ.

Analogously, we can define the notion of "δ-nice at the end" and we say that the configuration is δ-nice if it is δ-nice at the beginning and at the end.

Suppose now that B^1 and B^2 are two independent Brownian upcrossings of $\mathcal{A}(0, n)$. Note that when the configuration is δ-nice, then one can find a subarc of length at least δ on $\mathcal{C}_0 \cap \partial O^1_n$ that satisfies the conditions of Lemma 2.2. Also, O^1_n satisfies the conditions of Lemma 2.1. We shall use this later on.

Let

$$b^\delta_n = n^{-2}\mathbf{E}[e^{-\lambda L^1_n} 1_{\delta-\text{nice}}].$$

Lemma 4.1. *For every $\epsilon > 0$, there is a $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$,*

$$\tilde{b}_n - b^\delta_n \le \epsilon b_{n-2}.$$

Proof. Let $V = V_{n,\delta}$ be the event that the configuration is not δ-nice at the beginning, and let $U = U_{n,\delta}$ be the $U(\delta)$ as above. By symmetry and the time-reversal property of upcrossings, it suffices to show that for all δ sufficiently small,

$$n^{-2}\mathbf{E}[e^{-\lambda L^1_n} 1_V] \le \frac{\epsilon}{2} b_{n-2}.$$

Note that $V \cap \{L^1_n < \infty\} \subset U^c \cup V_1$ where

$$V_1 = \bigcup_{j=1}^{2} \{B[^j T^j_1, T^j_n] \cap \mathcal{A}(0, 4\delta) \neq \emptyset\}.$$

The strong Markov property, decompositions of Brownian upcrossings and monotonicity of extremal distance, combined with (4.3) imply that

$$n^{-2}\mathbf{E}[e^{-\lambda L^1_n} 1_{U^c}] \le c\mathbf{P}(U^c) b_{n-2}.$$

On the other hand, Lemma 3.3 establishes that

$$n^{-2}\mathbf{E}[e^{-\lambda L^1_n} 1_{V_1}] \le c\delta b_{n-1} \le c\delta b_{n-2}. \quad \square$$

Corollary 4.2. *For all δ sufficiently small, (4.2) holds.*

Proof. First, we claim that for all n sufficiently large

$$\#\{j \in \{1, \dots, n\} : b_{j+2} \ge c b_j\} \ge .9n, \tag{4.4}$$

where $c = e^{-80(2+\lambda_0)}$. To see this, assume not. Then, for infinitely many n's, there exists at least $.05n$ exceptional even values or at least $.05n$ exceptional odd values j in $\{1, \cdots, n\}$ scuh that $b_{j+2} \le c b_j$ in which case

$$b_{n+2} \le e^{-80(2+\lambda_0)(.05n)} \le e^{-2(2+\lambda_0)n} \le e^{-2\xi(2,\lambda)n}$$

and this contradicts the lower bound on b_{n+2}. By changing the value of c, we can conclude that (4.4) in fact holds for all $n \ge 1$. Hence, Lemma 4.1 (for $\varepsilon = c/4$) implies that for all δ sufficiently small, at least 90% of the integers j in $\{1, \dots, n\}$,

$$\tilde{b}_j - b^\delta_j \le c b_{j-2}/4 \le b_j/4 \le \tilde{b}_j/2$$

so that $b^\delta_j \ge \tilde{b}_j/2$. $\square$

4.2 Pasting

The goal is now to paste together nice configurations in order to get a lower bound for b^δ_{n+m+2} in terms of b^δ_n and b^δ_m. In order to do this, we will define "very nice configurations".

From now on, we fix a small value of δ such that (4.2) holds. We say that a configuration of pairs of upcrossings (B^1, B^2) of $\mathcal{A}(r, r')$ is "very nice at the end" if

- $L^1 < \infty$;

- $B^j(T^j_{r'-(1/3)}, T^j_{r'}) \subset \mathcal{A}(r'-(1/2), r'), \quad j = 1, 2.$
- $B^1 \cap \mathcal{A}(r' - \frac{1}{5}, r') \subset \{z : -\frac{1}{10} \le \arg(z) \le \frac{1}{10}\}$;
- $B^2 \cap \mathcal{A}(r' - \frac{1}{5}, r') \subset \{z : -\frac{1}{10} \le |\arg(z) - \pi| \le \frac{1}{10}\}$.
- $|\arg(B^1(T^1_{r'}))| \le 1/20$, $|\arg(B^2(T^2_{r'})) - \pi| \le 1/20$.

Note that there is no δ in this definition. Let

$$\beta^\delta_n = n^{-2}\mathbf{E}[\exp\{-\lambda L^1_n\} 1_{\delta\text{−nice at the beginning and very nice at the end}}].$$

However, by symmetry, the expectation is the same if we require the configuration to be δ-nice at the end and "very nice at the beginning." The goal is to paste together some configurations in $\mathcal{A}(0, n+1)$ that are "very nice at the end" with configurations in $\mathcal{A}(n+1, n+m+2)$ that are "very nice at the beginning."

Suppose that z^1 and z^2 are on $\mathcal{C}_{n+1}$, and let us now define $\beta^\delta_{n+1}(z^1, z^2)$ just as β^δ_{n+1} except that the upcrossings are conditionned to end at z^1 and z^2. In particular, since the law of the endpoints are uniform on $\mathcal{C}_{n+1}$, β^δ_{n+1} is the mean of $\beta^\delta_{n+1}(z^1, z^2)$, when z^1 and z^2 are integrated over $\mathcal{C}_{n+1} \times \mathcal{C}_{n+1}$. Note that $\beta^\delta_{n+1}(z^1, z^2) = 0$ as soon as $(z^1, z^2) \notin Q := \{e^{n+1+i\theta} : |\theta| < 1/20\} \times \{e^{n+1+i\theta} : |\theta - \pi| < 1/20\}$.

If $\alpha \in (0, \pi)$, the probability that a complex Brownian motion starting at $\epsilon \in (0, 1)$ reaches the unit circle without leaving the wedge $\{z : |\arg(z)| \le \alpha\}$ is at least $\epsilon^{\pi/(2\alpha)}$ (this is easy for $\alpha = \pi/2$ and can be established for other α by considering the map $z \mapsto z^{\pi/(2\alpha)}$). Such considerations show easily that if the configuration of upcrossings of $\mathcal{A}(0, n)$ is δ-nice, then with probability at least $c'\delta^c$, one can extend the upcrossings up to the circle $\mathcal{C}_{n+1}$ in such a way that the extensions first remain in different wedges (and also leave an empty wedge between them), that all the wedges intersect $\mathcal{A}(n-1, n)$ only inside the disks of radius δ around the points $B^j(T^j_n)$ and such that the obtained configuration of upcrossings of $\mathcal{A}(0, n+1)$ is very nice at the end. Furthermore, Lemmas 2.1, 2.3 and 2.2 show that we can also impose that $e^{-L^1_{n+1}} \ge c' e^{-L^1_n} \delta^c$. Finally, note that the weighted densities of the endpoints (on these configurations) on $\mathcal{C}_{n+1}$ are bounded away from zero on Q. Combining all this, we get that for any $(z^1, z^2) \in Q$,

$$\beta^\delta_{n+1}(z^1, z^2) \ge c'\delta^c b^\delta_n = c'' b^\delta_n \tag{4.5}$$

(recall that δ is fixed). Now we consider Brownian upcrossings B^1 and B^2 of $\mathcal{A}(0, n+1+1+n')$ that are decomposed as follows: A Brownian upcrossing of $\mathcal{A}(0, n+1)$, an intermediate part and a final Brownian upcrossing of $\mathcal{A}(n+1, n+1+1+n')$. By restricting ourselves only to the cases where the first parts create a δ-nice configuration at the beginning and are very nice at the end, where the intermediate parts are of diameter smaller than $\delta e^{n+1}/10$ and where the final parts are very nice at the beginning and δ-nice at the end, using Lemmas 2.1, 2.3 and 2.2 again, we get that

$$b^{\delta}_{n+n'+2} \geq c b^{\delta}_n b^{\delta}_{n'}$$

for some $c(\delta)$ (we omit the details here). This establishes (4.1) and finishes the proof of the upper bound. $\square$

5 Non-intersection probabilities

We now show how the preceeding results (and in particular the strong approximation for b^{δ}_n) can be used to derive Theorem 1.1, and "up-to-constants estimates" for other quantities closely related to a_n and b_n.

Proof of Theorem 1.1. Let $\mathcal{B}^j_r = Y^j[S^j_r, T^j_r]$ denote the traces of the upcrossings. For the upper bound, it suffices for example to remark that

$$\begin{aligned} Z_r(\mathcal{Y}^1_r, \mathcal{Y}^2_r) \leq \; & \mathbf{P}[\mathcal{B}^0_r \cap (\mathcal{B}^1_r \cup \mathcal{B}^2_r) = \emptyset \mid \mathcal{B}^1, \mathcal{B}^2] \\ & \times \mathbf{P}[Y^0[T^0_0, S^0_r] \text{ does not disconnect } \mathcal{C}_0 \text{ from infinity}] \\ & \times 1_{Y^1[T^1_0, S^1_r] \text{ and } Y^2[T^2_0, S^2_r] \text{ do not disconnect } \mathcal{C}_0 \text{ from } \mathcal{C}_r}. \end{aligned}$$

The first term is bounded above by cre^{-L}. The second term is bounded by a constant times $1/r$. The last event is independent of L and has probability bounded by cr^{-2}. Therefore $\mathbf{E}[Z^{\lambda}_r] \leq cb_r$.

For the lower bound, it suffices to use the lower bound for b^{δ}_n and to realize the Brownian paths $\mathcal{Y}^j$'s using a Brownian crossing of the annulus together with initial parts $Y^j[T^j_0, S^j_n]$ of small diameter. $\square$

Note that the estimates (3.1) and (3.5) of [10] follow similarly. Analogously, one can derive up-to-constants estimates if we prescribe the starting points of Y^1, Y^2 and/or of Y^0 on the unit circle. For instance, if we define

$$\hat{Z}_n(\mathcal{Y}^1_n, \mathcal{Y}^2_n) = \sup_{z \in \mathcal{C}_0} \mathbf{P}[\mathcal{Y}^0_n \cap (\mathcal{Y}^1_n \cup \mathcal{Y}^2_n) = \emptyset \mid Y^0(T^0_0) = z, \mathcal{Y}^1_n, \mathcal{Y}^2_n]$$

and

$$\hat{a}_n = \sup_{z_1, z_2 \in \mathcal{C}_0} \mathbf{E}[(\hat{Z}_n)^\lambda \mid Y^1(T_0^1) = z_1, Y^2(T_0^2) = z_2],$$

Then $a_n \leq \hat{a}_n$ and a simple application of the strong Markov property shows that $\hat{a}_n \leq c a_{n-1}$. In particular,

$$c_1' e^{-n\xi(2,\lambda)} \leq \hat{a}_n \leq c_2' e^{-n\xi(2,\lambda)},$$

for appropriately chosen c_1', c_2'.

6 Separation lemma

In this section, we prove an important lemma that states that no matter how bad O_n^1 is, then there is a good chance (with respect to the normalized measure weighted by $\exp(-\lambda L_{n+1}^1)$) that O_{n+1}^1 is very nice at the end as defined in Section 4.2. This lemma was the starting point for previous proofs of 'up-to-constants' estimates, see [4]. While we do not need this lemma to establish the estimates in this paper, we do use the lemma in [10] to prove analyticity of $\lambda \mapsto \xi(2, \lambda)$ (which was used to determine the disconnection exponents). For this reason, we include a proof here.

We use the notation of Section 4.2. We suppose that the upcrossings B^1, B^2 of $\mathcal{A}(0, r)$ are defined in a compatible way in terms of Bessel processes i.e., that both B^1 and B^2 are defined up to infinite time and that the upcrossings $B^1(0, T_r^1)$ and $B^2(0, T_r^2)$ define the configuration at radius e^r (O_r^1 and L_r^1 are then defined in terms of these configurations). $\mathcal{F}_r$ will denote the σ-field generated by these two paths. Recall that for all $r' > r$, conditionaly on $\mathcal{F}_r$, the law of $B^j[T_r^j, T_{r'}^j]$ is that of a Brownian motion started from $B^j(T_r^j)$ conditioned to hit $\mathcal{C}_{r'}$ before $\mathcal{C}_0$.

Define the event $\Delta(r, \delta)$ that the configuration in $\mathcal{A}(0, r)$ in δ-nice at the end, and the event G_r that it is very nice at the end.

Lemma 6.1 (Separation Lemma). *There exists $c > 0$ such that for all $n \geq 1$, for all $\lambda \in (0, \lambda_0]$,*

$$\mathbf{E}[1_{G_{n+1}} e^{-\lambda L_{n+1}^1} \mid \mathcal{F}_n] \geq c \mathbf{E}[e^{-\lambda L_{n+1}^1} \mid \mathcal{F}_n]. \tag{6.1}$$

Proof. We start by noting that estimates for Brownian motion in wedges show, just as for (4.5), that there exist c, c' such that for any 'stopping radius'

τ (i.e., stopping time for the filtration $(\mathcal{F}_s)_{s\geq 0}$), such that $\tau \in [n, n+1/4]$ almost surely,

$$\mathbf{E}[1_{G_{n+1}} e^{-\lambda L^1_{n+1}} \mid \mathcal{F}_\tau] \geq c' \delta^c e^{-\lambda L^1_\tau} 1_{\Delta(\tau,\delta)} \tag{6.2}$$

(because if the configuration is δ-nice at radius τ, then one can extend it in such way that it is very nice at radius $n+1$). Hence it suffices to find δ_0, c'' and such a stopping radius τ such that

$$\mathbf{E}[e^{-\lambda L^1_\tau} 1_{\Delta(\tau,\delta_0)} \mid \mathcal{F}_n] \geq c'' \mathbf{E}[e^{-\lambda L^1_\tau} \mid \mathcal{F}_n]. \tag{6.3}$$

For any positive integer m, let

$$\tau_m = \inf\{s \geq 0 : L^1_{n+s} = \infty \text{ or } \Delta(n+s, 2^{-m})\}.$$

Note that if $L^1_n < \infty$, then (up to a set of zero probability) $\tau_l = 0$ for all large enough l.

From the definition of δ-nice configurations, it is not difficult to see that there exists m_0 and $\rho > 0$ such that for all $m \geq m_0$,

$$\mathbf{P}[\tau_m \leq 2^{-m/20} \mid \mathcal{F}_n] \geq \rho.$$

By iterating, we see that

$$\mathbf{P}[\tau_m \geq m^2 2^{-m/20} \mid \mathcal{F}_n] \leq e^{-am^2},$$

for some positive constant a, and hence for all $m \geq m_0$,

$$\mathbf{E}[e^{-\lambda L^1_{\tau_m}} 1_{\tau_m \geq m^2 2^{-m/20}} \mid \mathcal{F}_n] \leq e^{-am^2} e^{-\lambda L^1_n}.$$

On the other hand, using estimates in wedges again, we see that for $m \geq m_0$,

$$\mathbf{E}[e^{-\lambda L^1_{\tau_m}} \mid \mathcal{F}_n] \geq c 2^{-a'm} 1_{\Delta(n, 2^{-m-1})} e^{-\lambda L^1_n}$$

so that there is a summable sequence $\{h_m\}$ such that

$$1_{\Delta(n,2^{-(m+1)})} \mathbf{E}[e^{-\lambda L^1_{\tau_m}} 1_{\tau_m \geq m^2 2^{-m/20}} \mid \mathcal{F}_n] \leq h_m \mathbf{E}[e^{-\lambda L^1_{\tau_m}} \mid \mathcal{F}_n].$$

Similarly (starting at radius $n + \tau_{m+1}$ instead of n),

$$\mathbf{E}[e^{-\lambda L^1_{\tau_m}} 1_{\tau_m \leq r(m)} \mid \mathcal{F}_{n+\tau_{m+1}}] \geq (1-h_m) \mathbf{E}[e^{-\lambda L^1_{\tau_m}} \mid \mathcal{F}_{n+\tau_{m+1}}]\, 1_{\tau_{m+1} \leq r(m+1)},$$

where $r(m) = \sum_{l=m}^\infty l^2 2^{-l/20}$. If we let m be the smallest integer such that $r(m) < 1/4$ and $h_l < 1$ for all $l \geq m$, then we get (6.3) with $\tau = n + (\tau_m \wedge 1/4)$, $\delta_0 = 2^{-m}$ and $c'' = \prod_{l=m}^\infty (1-h_l)$. $\square$

If $1 \leq n \leq m$, let

$$R_{n,m} = e^{(m-n)\xi}\mathbf{E}[e^{-\lambda L_m} \mid \mathcal{F}_n] \text{ and } R^*_{n,m} = e^{(m-n)\xi}\mathbf{E}[e^{-\lambda L_m} 1_{G_m} \mid \mathcal{F}_n].$$

Then, it follows from the lemma that there exists constants c_5, c_6 such that for all $m \geq n+1$,

$$R^*_{n,m} \leq R_{n,m} \leq c_6 R^*_{n,m},$$

$$c_5 R^*_{n,n+1} \leq R_{n,m} \leq c_6 R^*_{n,n+1}. \tag{6.4}$$

This result is used in [10].

In [4, 10] it is in fact shown that the limit $R_n = \lim_{m\to\infty} R_{n,m}$ exists and that

$$R_{n,m} = R_n[1 + \epsilon_{n,m}],$$

where $|\epsilon_{n,m}| \leq c_1 e^{-mc_2}$ and c_1, c_2 depend only on λ_0. Also, the limit

$$r = r(\lambda) = \lim_{n\to\infty} e^{n\xi(2,\lambda)} b_n$$

exists and

$$b_n = r e^{-n\xi(2,\lambda)}[1 + \epsilon_n]$$

where $|\epsilon_n| \leq c_1 e^{-mc_2}$.

7 Other exponents and exact values

The exponents $\xi(2, \lambda)$ comprise just one family of Brownian intersection exponents. The proofs apply with minor modifications to these other exponents. We review the results here.

Let $\bar{p} = (p_1, \ldots, p_l)$ be an l-tuple of positive integers and let $\bar{\lambda} = (\lambda_1, \ldots, \lambda_l)$ be an l-tuple of positive real numbers. Let

$$Y^{j,k}_t, \quad j = 1, 2, \ldots, l, \quad k = 1, 2, \ldots, p_j,$$

be independent Brownian motions starting uniformly on $\mathcal{C}_0$. As before, let

$$T^{j,k}_n = \inf\{t > 0 : Y^{j,k}_t \in \mathcal{C}_n\}.$$

For any $j = 1, \dots, l$, define

$$\mathcal{P}_n^j = \bigcup_{k=1}^{p_j} Y^{j,k}[0, T_n^{j,l}].$$

Let $E_{n,\bar{p}}$ be the event that the l packets of Brownian motions $\mathcal{P}_n^1, \dots, \mathcal{P}_n^l$ are disjoint and are ordered clockwise around the origin (i.e., that their intersection with $\mathcal{C}_n$ are ordered clockwise on $\mathcal{C}_n$). For each $k = 1, \dots, l$, let $Z_n^k = Z_n^k(\mathcal{P}_n^1, \dots, \mathcal{P}_n^l)$ denote the probability that a Brownian motion Y started uniformly on the unit circle reaches $\mathcal{C}_n$ without intersecting $\cup_{j=1}^l \mathcal{P}_n^j$, and in such a way that the endpoint of Y, $\mathcal{C}_n \cap \mathcal{P}_n^k$ and $\mathcal{C}_n \cap \mathcal{P}_n^{k-1}$ are ordered clockwise on the $\mathcal{C}_n$ (where $\mathcal{P}_n^0 = \mathcal{P}_n^l$). We then define

$$b_n(\lambda_1, p_1, \lambda_2, \dots, p_l) = \mathbf{E}[1_{E_{n,\bar{p}}} \prod_{j=1}^{l} (Z_n^j)^{\lambda_j}].$$

Theorem 7.1. *For every finite integers M and l, there exist constants $0 < c_1 < c_2 < \infty$ such that the following is true. For all positive integers $p_1, \dots, p_l$ that are smaller than M, for all positive reals $\lambda_1, \dots, \lambda_l$ that are smaller than M, there exists $\xi = \xi(\lambda_1, p_1, \dots, \lambda_l, p_l)$ such that for all $n \geq 1$,*

$$c_1 e^{-\xi n} \leq b_n(\lambda_1, p_1, \dots, p_l) \leq c_2 e^{-\xi n}.$$

Note (see [5]) that $\xi(\lambda_1, p_1, \lambda_2, \dots, \lambda_l, p_l)$ is unchanged if we change the order of the p's and the λ's. Hence, all b_n's (for different orderings of the p's and the λ's) are multiplicative constants away from each other.

There are also other exponents called the half-space exponents (see [5] for a precise definition). The methods of the present paper apply also for these exponents. We leave the detailed statement to the interested reader.

Nowhere in this paper have we used the exact values of the exponents. Rigorous determination of these values is the subject of the papers [7, 8, 9, 10]. In those papers we prove that

$$\xi(p_1, \lambda_1, \dots, p_l, \lambda_l) = V[U(p_1) + U(\lambda_1) + \cdots + U(\lambda_l)],$$

where

$$U(x) = \frac{\sqrt{24x+1} - 1}{\sqrt{24}} \text{ and } V(x) = \frac{6x^2 - 1}{12}.$$

In particular,

$$\xi(2, \lambda) = \frac{\lambda}{2} + \frac{11}{24} + \frac{5}{24}\sqrt{24\lambda + 1}.$$

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